An essay toward functional history.

***Calculus pedagogy in 2011*: suffering vascularized *off-skin* views of minding**

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1. Introduction

This article is a short *skirmish* toward future functional history. I look to issues and *issuings* in and around calculus pedagogy, and so draw attention to influences leading to present views on teaching calculus. The article is both short and a *skirmish*. It is short because it is a beginning exercise, toward learning how to do *functional history*. The community of course is not yet working functionally, so I move along here without, for example, being able to appeal to explanatory functional interpretations of statements about calculus or calculus pedagogy. In a few places, though, I point out where functional interpretations would be (will be) useful. The article also is a *skirmish*: The battling here is brief; but also relevant is the Sanskrit origin of the word, which means ‘skin’. And as the work below points to, a problem in past and present calculus pedagogy is an ongoing *oversight of insight*, a conceptualist tendency to hold one’s understanding *off-skin*, or rather, *off-chemo-skin* (a view that is contrary to the verifiable dynamics of knowing[1], inconsistent with any *basic position*[2]). The *oversight of insight* circulating within the community “organism”, has been skewing development and generating unrest, fear and dis-ease in otherwise talented students and dedicated teachers - both groups struggling with contemporary textbooks and mandated curricula. I end the article with some comments toward future cyclings of functional history.

2. A problem comes to light

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[1] See, e.g., B. Lonergan, Phenomenology and Logic: The Boston Lectures on Mathematical Logic and Existentialism, Collected Works of Lonergan, Vol. 18, University of Toronto Press, 2001, Appendix A.
[2] B. Lonergan, Insight, Collected Works of Bernard Lonergan, Vol. 3, University of Toronto Press, 2000, p. 413. (Below, I use the abbreviation, CWL3.)

In addition to those who "do calculus", there are reflections of mathematicians and teachers, the work of scholars in mathematics-education-for-calculus, the efforts and expressions of students of calculus, the work of education councils, and so on. Individuals from all of these groups have in various ways been contributing to calculus. As an ambient[3] context then, I invite your attention to our full calculus community, our body of experience that includes our past and present thought and talk about calculus, as well as our past and present thought and talk about our thought and talk about calculus.[4]

There are, for example, the nineteenth century developments leading to the elegance and rigor of delta-epsilon proofs. (These are now taught in many undergraduate programs in courses with such names as "advanced calculus", "introduction to analysis", or in the case of Michael Spivak, his landmark text simply called *Calculus*[5].) And from the beginnings of calculus [that is, from the work of Newton (1642-1727) and Leibniz (1646-1716) onwards] there have been applications of calculus formulas and techniques to "real world problems" in science, engineering and technologies. There has been interest in how best to teach calculus. There has been philosophic interest in calculus. And so on.

A problem for *functional history of calculus* comes to light when we begin to take notice of certain puzzling facts. Even though the subject has a secure scientific lineage[6] of more than three hundred years, and is now a normal part of science-and-technology curricula around the world, there is an ongoing unrest in the community, centered around the teaching and learning of calculus. Students have trouble with, and often fear standard presentations of the subject. Talented teachers struggle with available textbooks and mandated curricula structures. At another level, there are diverse and often opposed philosophies in mathematics and mathematics education[7] – about what calculus is, about what learning calculus is, about how best to teach

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[3] Containing and going-forward context.

[4] See, "generalized empirical method", B. Lonergan, Third Collection, p. 141.See also "GEM2", P. McShane, Field Nocturnes CanTower 44, The Fourth Stage of Meaning, http://www.philipmcshane.ca/FNC-44.pdf

[5] M. Spivak, Calculus, 3rd Ed., Cambridge University Press, 2006.

[6] Calculus emerged more than three hundred years ago with the work of Newton (1642-1727) and Leibniz (1646-1716). See, e.g., The Beginnings of Calculus, Ch. 12, in V. J. Katz, A History of Mathematics – An Introduction, 2nd Ed., Addison-Wesley-Longman Inc., Massachusetts, 1998.

[7] In the American setting, many competing voices in mathematics education are pointed to in David Klein's article, A Brief History of American K-12 Mathematics Education in the 20th Century, in

calculus. Related to this philosophic diversity, there has been an ongoing multiplication of "alternative pedagogies" for calculus, some compatible with each other, some not. Taken together, these observations lead to various questions: Why this ongoing unrest in the community? Why the fear? Why the multiplication of teaching strategies? What has been, and is, going on in the calculus community[8]? These are large questions regarding complex combinations of (genetic) sequences of developments within the community, and this is a short *skirmish* of an article. Still, it turns out that by appealing to available documents and data of consciousness, we can uncover and point to something that evidently has been contributing cumulatively to our present difficulties.

3. Pointings toward historical data – preliminary gatherings and samplings

Let's start by thinking back to the early days of calculus, the (independent) discoveries of both Newton (1642-1727) and Leibniz (1646-1716). Future functional history will rely on functional interpretations; and functional interpretations (will, in explanatory detail) reveal[9] that both Newton and Leibniz had genetically comparable initial key insights. In both cases, their work was partly explanatory with, though, some elements left described or undefined. For instance, for Newton, continuously varying motion was "intuitive"[10]. But in ratios of change, by putting "all the products equal to nothing" he discovered what he called the *fluxion*, "speed with which *x*

Mathematical Cognition: A Volume in Current Perspectives on Cognition, Learning, and Instruction. Mass., Information Age Publishing, Inc., 2003, p175- 225. "Abstract: This chapter describes and analyzes the major conflicts over K-12 mathematics education that erupted among professional educators, psychologists, mathematicians, and parents of school children in the U.S. during the 20th century. Political struggles and policy changes in mathematics education in the 1980s and the 1990s are given special attention."

[8] And of course, ultimately within the whole body that is our global historical community of mathematics, sciences, cultures, arts, technologies, economies. See Section 5 below.

[9] In this exercise toward functional history, I point to (future) functional interpretations of, e.g., the results of Newton and Leibniz. Future functional interpretations will reach for explanation within a "generalized empirical method" (B. Lonergan, Third Collection, Paulist Press, New York and G. Chapman, London, 1985, p. 141). Explanatory interpretation will therefore include genetic and dialectical sequences and series, of genera and species of systems of insights, within the full human aggreformic heuristic of the *metagrams W1* and *W2*. Regarding *metagrams*, see P. McShane, Prehumous 2, Metagrams and Metaphysics, http://www.philipmcshane.ca/prehumous-02.pdf. Leads on the possibility of explanatory interpretation can be found in Insight, CWL3, Sec.17.3 and Method in Theology, Darton, Longman & Todd, 1975, Ch. 7. A reader not yet in possession of the key calculus insights may find help in the article: T. Quinn: "The Calculus Campaign", Journal of Macrodynamic Analysis, 2 (2002): 8-36, *http://www.mun.ca/jmda/vol2/calculus.pdf.*

[10] V. J. Katz, A History of Mathematics – An Introduction, 2nd Ed., Addison-Wesley-Longman Inc., Massachusetts, 1998, p. 510.

increased through its generating motion", expressed by dot overscripts[11]. Leibniz used a different notation in his work, differentials *dx*, *dy* ("arbitrary finite line segments" which he did not rigorously define).[12] Unlike the dot notation of Newton, Leibniz's notation more easily led to new results, e.g., the product rule: "*d*(*xy*) is the same as the difference between two successive *xy's*; let one of these by *xy*, and the other $(x + dx)(y + dy)$; then we have $d(xy) = (x + dx)(y + dy) – xy = xdy + ydx +dxdy$. The omission of the quantity *dxdy*, which is infinitely small in comparison with the rest, … will leave $xdy+ y\ dx$".[13] The Leibniz notation also made it especially convenient for discovering the differential of an area obtained from "a sum of rectangles of differential width *dx*". As is well known, this is now called the "the fundamental theorem of calculus" - which, using the Leibniz stylized 'S' for "sum", is $d(\int ydx) = ydx$.[14]

Following the initial breakthroughs of Newton and Leibniz, early calculus textbooks began to appear, some using Newton's terminology, and others that of Leibniz.[15] Cauchy (1789-1857) though "was not satisfied with what he believed were unfounded manipulations of algebraic expressions, especially infinitely long ones."[16] He broke though to a solution with his definition of what he called a *limit*.[17] After Cauchy, we go beyond "mere calculus" and see the impressive rise of *real analysis* – thanks to advances made by Fourier (1768-1830), Dirichlet (1805-1859), Weierstrass (1815-1897), Heine (1821-1881), Cantor (1845-1918), and others. These mathematical developments though go well beyond calculus and so I will not discuss them directly in this short article.

Influenced in various ways by (i) advances made in real analysis; (ii) traditions of axiomatic presentation (going back, e.g., to Euclid and Archimedes); (iii) emerging applications in engineering and the sciences; and (iv) increasing interest in "education for all", calculus textbooks started becoming available for different kinds of 20th century audience (professional

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[11] See, e.g., V.J. Katz, A History of Mathematics – An Introduction, 2nd Ed., Addison-Wesley, 1998, p. 510.
[12] "He was reluctant to define his differentials *dx* as 'infinitesimals' because he believed there would be great of criticism of these quantities which had not been rigorously defined." Ibid, p. 527.
[13] This is a translation from a Leibniz manuscript, taken from V. J. Katz, Ibid, also p. 527.
[14] Op. cit., 524.
[15] Ibid, pp. 532-535.
[16] Ibid., p. 707.
[17] The definition occurs at the beginning of his 1821 book, *Cours d'Analyse de l'École Royale Polytechnique*.

mathematicians, teachers, students, general public). There also was the 20th century emergence of the discipline now called Mathematics Education, and organizations such as the MAA (Mathematical Association of America[18]), and the NCTM (National Council of Teachers of Mathematics, since 1921). And in the late 20th century there was the Harvard Calculus Reform[19]. All of these have, to various degrees, been influential, so let's have a brief look at what these groups and standard calculus textbooks say about teaching calculus.

The Executive Summary of the NCTM[20] includes the following statement under *Learning*: "By aligning factual knowledge and procedural proficiency with conceptual knowledge, students can become effective learners." Later in the NCTM Summary (under Standards) we find: "Algebra is best learned as a set of concepts and techniques tied to the representation of quantitative relations and as a style of mathematical thinking for formalizing patterns, functions, and generalizations." While algebra is not calculus, algebra is part of calculus, and NCTM Principles and Standards have in some cases been incorporated into foundational views grounding studies of high school calculus pedagogy. See, for example, "Developing Student Understanding: Contextualizing Calculus Concepts".[21] The authors' abstract includes: "The qualitative study sought to describe several critical aspects of understanding: students' ability to explain concepts and procedures, to apply concepts in a physics context, and to explore their own learning. … . This study suggests that making connections between calculus and physics can yield deep understandings of semantic as well as procedural knowledge."

Now, let's also look to a few statements from two of the standard university calculus books in use at this time. The first is a 5th edition textbook in world-wide publication[22]. Under Textbook Features of the book we find: Capstones: … exercises synthesize the main concepts …; Writing About Concepts: … understanding the basic concepts …; Examples: … examples are worked out step-by-step. These worked examples demonstrate the procedures and techniques for solving

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[18] The MAA was founded in 1915 and is headquartered at 1529 18th Street, Northwest, Dupont Circle, Washington, D.C..
[19] Calculus Consortium at Harvard. (1994). Calculus. New York: John Wiley and Sons.
[20] http://www.nctm.org/standards/content.aspx?id=11608
[21] Schwalbach, E. M. and Dosemagen, D. M. (2000), Developing Student Understanding: Contextualizing Calculus Concepts, School Science and Mathematics, 100: 90–98.
[22] R. Larson and B. H. Edwards, Calculus – Early Transcendentals, 5th Ed., Brooks/Cole, Cengage Learning, 2011. This book is at hand. The features I point to though are found in most textbooks in common use.

problems, and give students an increased understanding of the concepts of calculus. …; Review Exercises: … review of the chapter's concepts … an excellent way to prepare for an exam. … ; Theorems provide the conceptual framework for calculus …; Chapter Openers … provide initial motivation … an important concept in the chapter is related to an application of the topic in the real world. …; Explorations … provide students with … challenges to study concepts …; Historical Notes and Biographies … teach students about the people who contributed to its (calculus) formal creation. …; Technology … used to help … explore the concepts of calculus." As is now normal in calculus books, there is a mainly axiomatically organized layout of chapters and content within sections. Chapter 2 on Limits and their Properties precedes Chapter 3 on Differentiation; Chapter 4 is then Applications of Differentiation; Chapter 5 on Integration introduces Riemann sums before the fundamental theorem of calculus; Chapter 6 on Differential Equations begins by recalling the *nominal definition*[23] of *differential equation* given earlier in Chapter 5 (p.285), and then moves toward sections on applications; Chapter 7 is Applications of Integration; Chapter 8 is Techniques of Integration; Chapter 9 is Infinite Series; and the last chapter on single variable calculus is Chapter 10 Conics, Parametric Equations and Polar Coordinates.

A second well known book is in its second edition[24]. Writing Exercises are said to "explain … concepts"[25]. Again, the order of presentation in the book is guided by axiomatic criteria and orderings of concepts. For example, Chapter 1 defines *function*, Chapter 2 defines *limit*, Section 3.1 defines *derivative*, Section 3.3. states the power rule for positive integer powers of x, followed by a proof of this general formula using a dropped in general factorization formula for a certain n^{th} degree polynomial.

In the 1990's, the "Harvard Calculus Reform" was an attempt to improve how calculus was taught. The Harvard Reform approach advocates the following "principles": 1. Mix a graphical, numerical and algebraic approach ("rule of three"[26]); 2. Motivate by practical problems ("the way

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[23] Lonergan, CWL3, p. 35.
[24] J. Hass, M. D. Weir, G. B. Thomas Jr., University Calculus – Early Transcendentals, 2nd Ed., Addison-Wesley, Boston, 2012.
[25] Ibid., p. xii.
[26] Later extended to the "rule of four": graphical, numeric, symbolic/algebraic, and verbal/applied presentations. See, e.g., the description of the 5th edition of the Calculus Reform Calculus text, quoted in Note 29, below.

of Archimedes"); 3. Choose topics which interact with other disciplines; 4. Formulate open ended word problems; 5. Discourage the mimic template techniques; 6. Use technology to visualize concepts; 7. Prefer plain English over formal descriptions.

At this time, mainstream Mathematics Education is grounded in "constructivism", the view of mathematical learning that students "construct understandings of mathematical concepts".[27] The 2008 MAA multi-author publication, Making the Connection[28], carries the constructivist view laced throughout. We may look, e.g., to the book's Table of Contents for a few signs of this: "grasping the concept of variable"; "To understand the idea of accumulation, students must first acquire a process view of formulae and a covariational concept of function"; "naïve notions of the concept of infinity"; "the concept of divisibility", "conceptually oriented learning"; "techniques for supporting students in understanding and using definitions"; "leverage computer technology to support students in building both inductive and deductive reasoning skills"; "uses of examples … for students' understanding of concepts"; and so on.

4. Toward future functional history of *calculus pedagogy*

Let's look again to the sources mentioned above - the textbooks, the Harvard Reform Principles, the quotations from Making the Connection. What is going on? Are there links or commonalities? What do they have in common? Is there some kind pattern within this dynamic body of work?

One commonality is that all of these sources implicitly or explicitly (or both) give primacy to "concept". In particular, for standard calculus texts the organization of chapters and topics is generally guided by axiomatic criteria and "ordered sets of concepts". [29] Discussions "introduce

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[27] Constructivism is a branch-line of *conceptualism*. One of the "parents" of conceptualism was J. Duns Scotus (1265/66-1308), advocating the view that having concepts precedes reaching understanding; and that understanding is a matter of connecting concepts.

[28] Making the Connection, Research and Teaching in Undergraduate Mathematics, Pub. By the MAA, 2008.

[29] Drawing on the experience and reflections of members of the consortium, the Harvard Reform descriptive teaching strategies edge up on normative needs (experience, …, the dynamics of knowing, CWL18, Appendix A). But, e.g., "Principle 6" points to a disorientation regarding "concepts". (See paragraphs immediately below on **McA** views of minding.) And this notion of "concept" is carried into the writing of the Reform *Calculus* textbook resulting from the Consortium's efforts: [John Wiley & Sons, 2004, "Calculus teachers recognize 'Calculus' as the leading resource among the 'reform' projects

concepts" first, follow by presentations for "understanding concepts"; later sections are on "applying concepts". Imposed on students and colleagues are, then, the consequences of a view of minding that one has concepts prior to reaching understanding - the **McA**[30] view of minding: **M**inding is having **c**oncepts followed **A**nalysis of concepts (connections made between concepts, concepts appended to each other, and so on).[31] This view of minding though runs radically counter to our verifiable dynamics of knowing[32] - the **MAC** view: "… '**M**' refers to Mind (your mind), '**A**' has the meaning of Ah? (What?) and Aha! (direct insight), and 'C' refers to concept, formulation, definition".[33]

This is a preliminary and merely descriptive identification, but historically significant. A psychoanalyst might make an initial breakthrough toward being able to explain a client's adult water phobia by uncovering a traumatic event from the client's childhood and consequently be able to describe cumulative skew effects of this event on the client's biography. But, reaching explanatory understanding would be a further major achievement for the psychoanalyst[34]. In the calculus teaching problem, we have reached a similar initial descriptive breakthrough, by noticing the occurrence of a kind of (usually silent) trauma - not isolated, but up-taken *throughout-throughin* the developments of our "community biography". Why *silent* trauma? The sad reason is simple: Implemented, the McA view very effectively blocks, re-directs and skews

that employ the rule of four and streamline the curriculum in order to deepen conceptual understanding. The fifth edition uses all strands of the "Rule of Four" - graphical, numeric, symbolic/algebraic, and verbal/applied presentations - to make concepts easier to understand. The book focuses on exploring fundamental ideas rather than comprehensive coverage of multiple similar cases that are not fundamentally unique. Readers will also gain access to WileyPLUS, an online tool that allows for extensive drills and practice. Calculus teachers will build on their understanding in the field and discover new ways to present concepts to their students."]

[30] P. McShane, John Benton, Allesandra Drage, Introducing Critical Thinking, Axial Press, 2005, pp. 64-65.

[31] As mentioned above in Note 27, a parent of the McA view was John Duns Scotus (1265/66-1308). Future functional interpretation of the work of Scotus will be useful here.

[32] See, e.g., B. Lonergan, Phenomenology and Logic, CWL18, Collected Works of Bernard Lonergan, Vol., 18, University of Toronto Press, 2001, Appendix A, pp. 319-321.

[33] See McShane, Benton, Drage, p, 65. So, we have M?!C as a verifiable pattern within the fuller dynamics of knowing pointed to in CWL18 (Appendix A), that includes Is? ! (reflective insight) J (judgment).

[34] See B. Lonergan, CWL3, Section 17.3; and Method in Theology, Darton, Longman & Todd, 1975, Ch. 7.

the “childhood” of any potential insight in calculus – insight that otherwise would generate our *inner-off-spring*, our inner words.[35]

At the community level, the McA view directs pedagogy away from incarnating the basic doctrine that “teaching children calculus is teaching children children”[36]. Attempting to teach “concepts first”, “conceptual understanding”, has instead been having a contrary effect: “teaching children calculus is teaching children con-*septa*[37].” Evidently, to teach as though we get concepts prior to understanding *fences* students off from their own experience, their own dynamics of growth, from their own wonder-chemo-*skin*, and indeed from their wonder-chemos-*kin* – that is, from self and from other selves, *self-screens and self-screenings* throughout-throughin.

5. Beyond calculus – comments to functional historians

I step back now from my attempt to edge into a poise of a *functional history skirmish* - of a skin-quest quest-skin of ‘What’s going on in calculus pedagogy?’ Because it is early days yet for functional collaboration, I now add a few comments toward future functional histories of mathematics. Also implicit are questions for functional dialectics.

Calculus emerged in the late 17th century, and continued to develop, leading to the rise of what are now called commutative and non-commutative modern analysis. Other magnificent developments have been occurring throughout mathematics. At the same time, there is a global and historical prevalence of the McA view of mathematical development[38], revealing that

[35] Witness, e.g., the (aggreformic: see W1 in P. McShane, Prehumous 2, Metagrams and Metaphysics, http://www.philipmcshane.ca/prehumous-02.pdf)) student turned off by McA techniques, or in some cases experiencing gut-felt fear --- when psychologically assaulted by names, charts, and symbols, prior to possession of appropriate data, prior to fostered wonder, prior to insight and inner formulation.
[36] “‘When teaching children geometry one is teaching children children.’ The slogan has, of course, more general forms: for geometry one can substitute any topic; for children one can substitute adult; and the adult can be oneself.” P. McShane, *Divyadaan* 13/3 (2002) 279-309, http://www.divyadaan.org/Journal/Journal.htm
[37] From Latin *saeptum*, enclosure, fence, wall, from *saepire* to fence in, from *saepes* fence, hedge.
[38] Constructivism is foundational to mainstream Mathematics Education at this time (2011). For an example beyond calculus, we can look to the book Learning Abstract Algebra with ISETL Learning (Springer-Verlag, New York, 1994), by Ed Dubinsky and Uri Leon, influential representatives of constructivism and reform efforts. The authors provide a section called Comments for the Student. Some of the statements are: “If you use this book properly, …, abstract mathematical concepts will start to make sense” (p. xi). “… in working with the computer … you will have constructed the meaning that the

mathematical development has not yet been luminously digested by the community[39]. But, the *oversight of insight* has been vascularized and has metastasized, influencing the flows and *fluxions* of flows of views, conceptions and implementations within the developing global community organism. There is, then, the need of broader reaching functional histories in mathematics within the body of global history.

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symbols represent". (p. xii). "You will write small pieces of code, or "programs" that get the computer to perform various mathematical operations. In getting the computer to work the mathematics, you will more or less automatically learn how the mathematics works! Anytime you construct something on a computer then, whether you know it or not, you are constructing something in your mind." (p. xii). "Writing definitions and proofs and solving mathematical problems is like writing programs in a mathematical programming language and executing them in your head" (p. xiii). "The nice thing about this (computer) language is that the way it works is very close to the way mathematics works." (p. xiii). In Comments for the Instructor we find: "the student is given considerable help in making mathematical constructions to use in make sense out of the material." (p. xvii) Evidently, the approach does not conform with mathematical experience; and except for a few outliers, mathematicians at least tacitly agree - for the book is not used by algebraists to teach abstract algebra.

[39] Possible through "generalized empirical method", B. Lonergan, Third Collection, p. 141; or GEM2 in P. McShane, Field Nocturnes CanTower 44, The Fourth Stage of Meaning, http://www.philipmcshane.ca/FNC-44.pdf